

A PATHOLOGY OF ASYMPTOTIC MULTIPLICITY IN THE RELATIVE SETTING

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ABSTRACT. We point out an example of a projective family $\pi : X \rightarrow S$, a π -pseudoeffective divisor D on X , and a subvariety $V \subset X$ for which the asymptotic multiplicity $\sigma_V(D; X/S)$ is infinite. This shows that the divisorial Zariski decomposition is not always defined for pseudoeffective divisors in the relative setting.

1. INTRODUCTION

Suppose that X is a smooth projective variety and D is a pseudoeffective \mathbb{R} -divisor on X . The asymptotic multiplicity of D along a subvariety $V \subset X$, studied by Nakayama [10] and Ein-Lazarsfeld-Mustață-Nakamaye-Popa [4], has proved to be a fundamental tool in understanding the properties of the divisor D . For big divisors D , the definition of the asymptotic multiplicity is straightforward: roughly, one considers the linear series $|mD|$ for larger and larger values of m , and takes $\sigma_V(D) = \lim_{m \rightarrow \infty} \frac{1}{m} \text{mult}_V |mD|$, where the multiplicity of a linear series along a subvariety is defined to be the multiplicity of a general member.

Complications arise, however, in carrying out this construction for divisors D which are pseudoeffective but not big, i.e. for divisors on the boundary of the pseudoeffective cone $\overline{\text{Eff}}(X) \subset N^1(X)_{\mathbb{R}}$. Nakayama realized that $\sigma_V(D)$ can be extended to a lower semicontinuous function on $\overline{\text{Eff}}(X)$ by setting

$$\sigma_V(D) = \lim_{\epsilon \rightarrow 0} \sigma_V(D + \epsilon A),$$

where A is a fixed ample divisor. In some applications (e.g. in the construction of Zariski decompositions), it is important to know that the limit in question takes a finite value. While it is clear that the quantity on the right is nondecreasing as ϵ is made smaller, it might *a priori* be unbounded in the limit. That this does not happen in the non-relative setting was observed by Nakayama.

Our aim in this note is to demonstrate by example that when asymptotic multiplicity invariants are considered in the greater generality of divisors on a projective family $\pi : X \rightarrow S$, this finiteness need not hold: for a π -pseudoeffective divisor, the limit defining $\sigma_V(D; X/S)$ can indeed be infinite. This answers a question of Nakayama [10, pg. 33]. The example itself is familiar, a divisor on the versal deformation space of a fiber of Kodaira type I_2 , which has been considered in related contexts by Reid [11, 6.8] and Kawamata [7, Example 3.8(2)], [8, Example 9].

Theorem 1. *There exists a projective family $\pi : X \rightarrow S$, a π -pseudoeffective divisor D , and a subvariety $V \subset X$ for which $\sigma_V(D; X/S)$ is infinite.*

An important use of asymptotic multiplicity invariants is in the construction of the divisorial Zariski decomposition, a higher-dimensional analog of the usual Zariski decomposition

on surfaces. The example here shows that trouble arises if one generalizes this construction to pseudoeffective classes in the relative setting: after passing to a blow-up on which the valuation corresponding to V is divisorial, we obtain an example in which the decomposition is not defined.

Corollary 2. *Let $\pi : X \rightarrow S$ be as in Theorem 1. If $f : W \rightarrow X$ is the blow-up along V with exceptional divisor E , then $\tilde{D} = f^*D$ has $\sigma_E(\tilde{D}; W/S) = \infty$ and $N_\sigma(\tilde{D}; W/S)$ is not defined.*

Moreover, the divisor \tilde{D} does not admit any Zariski decomposition in a very strong sense:

Corollary 3. *There does not exist a birational model $g : Z \rightarrow W$ for which $g^*\tilde{D}$ admits a decomposition $g^*\tilde{D} = P + N$ with P a $g \circ (f \circ \pi)$ -movable divisor and N effective.*

In Section 2 we recall the basic definitions and properties of the invariants $\sigma_V(D; X/S)$ and $N_\sigma(D; X/S)$ appearing in Theorem 1 and Corollary 2, before establishing the claims in Section 3. In Section 4, we describe a more general setting for making computations in a similar spirit.

2. PRELIMINARIES

Suppose that $\pi : X \rightarrow S$ is a projective, surjective morphism with connected fibers, with X and S normal and \mathbb{Q} -factorial (hereafter, a *nice family*). We will find it convenient to allow the base S to be a surface germ, following [6]. Two divisors D and D' on X are said to be numerically equivalent over S , or π -numerically equivalent, if $D \cdot C = D' \cdot C$ for any curve C that is contracted by π ; write $D \equiv_\pi D'$ for the relation of numerical equivalence over S , and $N^1(X/S)$ for the vector space of \mathbb{R} -divisors on X , modulo this equivalence.

The familiar cones of positive divisors on a projective variety all have analogs in the relative setting: a divisor D on X is said to be

- (1) π -ample if D_s is ample on every fiber $X_s = \pi^{-1}(s)$;
- (2) π -nef if D_s is nef on every fiber X_s (i.e. if $D \cdot C \geq 0$ for every curve C contracted by π);
- (3) π -movable if the support of the cokernel of $f^*f_*\mathcal{O}_X(D) \rightarrow \mathcal{O}_X(D)$ has codimension at least 2;
- (4) π -big if the restriction of D to the generic fiber is big;
- (5) π -pseudoeffective if the restriction of D to the generic fiber is pseudoeffective.

Corresponding to these classes of divisors are cones inside $N^1(X/S)$:

$$\text{Amp}(X/S) \subseteq \text{Mov}(X/S) \subseteq \overline{\text{Eff}}(X/S).$$

We note that the cone $\overline{\text{Eff}}(X/S)$ is not necessarily a strongly convex cone, in that it might contain an entire line through the origin; this contrasts with the familiar case when S is a point. For example, if D restricts to 0 on a general fiber of π , then D and $-D$ are both π -pseudoeffective.

For simplicity, we will assume that the base space S is affine and that there exists a π -ample divisor A on X . This is not really necessary, but the invariants under consideration can be computed in the general setting simply restricting to the preimage of a suitable affine open set; we refer to [10, §3.2] for details. If D is a π -big divisor, then $f_*\mathcal{O}_X(mD) \neq 0$ for sufficiently large and divisible m , and so $H^0(X, \mathcal{O}_X(mD)) = f_*(\mathcal{O}_X(mD))$ is nonzero as well. Hence if S is affine, any π -big class has an effective representative.

Definition 1. Given an irreducible subvariety $V \subset X$ and a π -big \mathbb{R} -divisor D , set

$$\sigma_V(D; X/S) = \inf_{\substack{D' \equiv_{\pi} D \\ D' \geq 0}} \text{mult}_V(D').$$

Since D is π -big, there exists an effective \mathbb{R} -divisor D' that is π -numerically equivalent to D , and this infimum is taken over a nonempty set.

In the definition, D' ranges over effective \mathbb{R} -divisors that are π -numerically equivalent to D . When $S = \text{Spec } \mathbb{C}$ and D is a big integral divisor, a sequence D'_m of such \mathbb{R} -divisors with multiplicities converging to the infimum can be found by taking $D'_m \in \frac{1}{m} |mD|$, where we choose a general element of the linear system $|mD|$.

We next extend the definition of the asymptotic multiplicity from π -big divisors to π -pseudoeffective divisors.

Definition 2. Given a π -pseudoeffective \mathbb{R} -divisor D , set

$$\sigma_V(D; X/S) = \lim_{\epsilon \rightarrow 0} \sigma_V(D + \epsilon A; X/S).$$

This is evidently a nondecreasing function as ϵ approaches 0, but it might have infinite limit. To show that it has a finite limit, it suffices to bound $\sigma_V(D + \epsilon A; X/S)$ above, independent of ϵ . Nakayama gives several conditions under which this can be achieved.

Theorem 4 ([10], Lemmas 2.1.2, 3.2.6). *If any of the following holds, then $\sigma_V(D; X/S)$ is finite.*

- (1) $S = \text{Spec } \mathbb{C}$ is a point;
- (2) D is numerically equivalent over S to an effective \mathbb{R} -divisor Δ ;
- (3) $\text{codim } \pi(V) < 2$.

We recall the proof in case (1), perhaps the most important in practice. Case (2) is immediate from the definition, and we refer to [10] for (3). Assume for a moment that $V \subset X$ is an irreducible divisor; that this implies the general statement will follow from Theorem 5(2) below.

Proof of (1). For any ϵ , $(D + \epsilon A) - \sigma_V(D + \epsilon A)V$ is pseudoeffective, and so

$$((D + \epsilon A) - \sigma_V(D + \epsilon A)V) \cdot A^{n-1} \geq 0.$$

As long as $\epsilon < 1$ it follows that

$$\sigma_V(D + \epsilon A) \leq \frac{(D + \epsilon A) \cdot A^{n-1}}{V \cdot A^{n-1}} \leq \frac{(D + A) \cdot A^{n-1}}{V \cdot A^{n-1}}$$

is bounded above as ϵ decreases to 0. □

This argument relies in a crucial way on the properness of X to carry out intersection theory, and is not applicable in the relative setting in general.

Proposition 5 ([10], Lemmas 2.1.4, 2.2.2, 2.1.7). *Suppose that $\pi : X \rightarrow S$ is a nice family and $V \subset X$ is an irreducible subvariety.*

- (1) *If F is any π -pseudoeffective divisor on X , then*

$$\lim_{\epsilon \rightarrow 0} \sigma_V(D + \epsilon F; X/S) = \sigma_V(D; X/S).$$

- (2) *Let $f : W \rightarrow X$ be the normalized blow-up of X along V , and let E be a component of the exceptional divisor over V . Then $\sigma_E(f^*D; W/S) = \sigma_V(D; X/S)$.*

(3) *The number of prime divisors Γ for which $\sigma_\Gamma(D; X/S) > 0$ is finite.*

The first of these shows that Definition 1 is independent of the choice of π -ample divisor A , while the second completes the proof of Theorem 5.

Definition 3 ([10], [3]). Suppose that $\pi : X \rightarrow S$ is a nice family and that D is a π -pseudoeffective divisor such that $\sigma_\Gamma(D; X/S)$ is finite for every prime divisor Γ . Then set

$$N_\sigma(D; X/S) = \sum_{\Gamma} \sigma_\Gamma(D; X/S) \Gamma,$$

$$P_\sigma(D; X/S) = D - N_\sigma(D; X/S).$$

It follows from Proposition 5(3) that there are only finitely many nonzero terms in the sum defining $N_\sigma(D; X/S)$.

We refer to $N_\sigma(D; X/S)$ as the negative part of the Zariski decomposition, and $P_\sigma(D; X/S)$ as the positive part. The negative part is a rigid, effective divisor. The positive part might not be nef, but it lies in the closure $\overline{\text{Mov}}(X/S)$ of the cone $\text{Mov}(X/S)$. Corollary 2 shows that without the finiteness hypothesis on $\sigma_\Gamma(D; X/S)$, the definition is not always applicable in the relative setting.

An equivalent approach to defining this decomposition is given by Kawamata via the *numerically fixed part* of a linear series [8].

Definition 4. Suppose that $\pi : X \rightarrow S$ is a nice family. Then

$$N_\sigma(D; X/S) = \lim_{\epsilon \rightarrow 0} (\inf D' : D' \equiv_\pi D + \epsilon A, D' \geq 0)$$

where the infimum of divisors is defined coefficient-wise.

In the non-relative setting, the divisorial Zariski decomposition is defined for any pseudoeffective class D , but it lacks certain useful properties of the classical Zariski decomposition in dimension 2. In particular, the failure of the positive part P to be nef can be problematic. It is often useful to try to construct a birational model $f : W \rightarrow X$ for which $P_\sigma(f^*D)$ is actually nef. This suggests that higher-dimensional versions of Zariski decomposition should allow passage to a higher birational model. There are several possible definitions, among them the weak Zariski decomposition of Birkar.

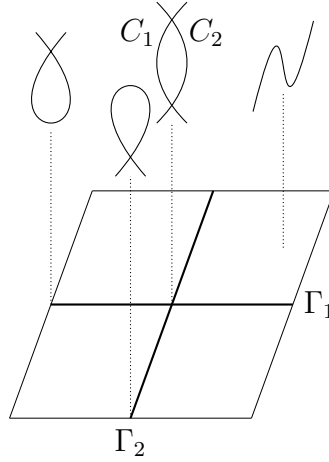
Definition 5 ([2]). Suppose that $\pi : X \rightarrow S$ is a nice family and that D is a pseudoeffective divisor on X . We say that D admits a *weak Zariski decomposition over S* if there exists a birational map $f : Y \rightarrow X$ and a decomposition $f^*D = P + N$, where P is π -nef and N is effective.

This condition is fairly unrestrictive in that it does not impose any analog of the negative-definiteness required in the two-dimensional setting. Nevertheless, there exist pseudoeffective \mathbb{R} -divisors on smooth threefolds which do not admit a weak Zariski decomposition [9]. Corollary 3 asserts that the divisor \tilde{D} provides another such example. Indeed, \tilde{D} admits no Zariski decomposition in a still stronger sense: even after pulling back to a higher model, it cannot be decomposed as the sum of an effective divisor and a relatively movable divisor. The example is qualitatively rather different from that of [9]: there, a certain pseudoeffective divisor D_λ has negative intersection with infinitely many curves; here, there is a single curve on which D is negative, but the multiplicity of D along this curve is infinite.

3. MAIN EXAMPLE

The claimed pathology follows from a few calculations on an example that has been studied by Kawamata and Reid. Let $\pi : X \rightarrow S$ be the versal deformation space of a fiber of Kodaira type I_2 . The base S is smooth, 2-dimensional germ. The fiber over the central point $0 \in S$ consists of two smooth rational curves C_1 and C_2 , meeting transversely at two points p_1 and p_2 . Let $C = \pi^{-1}(0)$ be the union of these two curves.

There are two divisors $\Gamma_1, \Gamma_2 \subset S$ corresponding to the smoothings of the two nodes of C . The fiber of π over a general point of Γ_i is a nodal rational curve, while the fiber over a general point of S is a smooth curve of genus 1.


 FIGURE 1. The family $\pi : X \rightarrow S$

Lemma 6. *The normal bundle $N_{C_i/X}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.*

Proof. Suppose for simplicity that $i = 1$. There is an exact sequence

$$0 \longrightarrow N_{C_1/X} \longrightarrow (N_{C/X})|_{C_1} \longrightarrow T_{C_2,p_1} \oplus T_{C_2,p_2} \longrightarrow 0$$

with the property that a first-order deformation, determined by a section $s \in H^0(C, N_{C/X})$ smooths the node at p_i if and only if s has nonzero image T_{C_2,p_i} [5, Lemma 2.6]. The sheaf in the middle is the trivial $\mathcal{O}_C \oplus \mathcal{O}_C$. In one direction p_1 is smoothed, and in another p_2 is, so the map sends $(1, 0)$ to $(1, 0)$ and $(0, 1)$ to $(0, 1)$ with respect to the direct sum decompositions. It follows that the kernel is $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. \square

Lemma 7. *There exists a flop $\tau : X \dashrightarrow X^+/S$ with flopping curve C_1 . Let $C_1^+ \subset X^+$ be the flopped curve, and $C_2' \subset X^+$ be the strict transform of C_2 . There exists an isomorphism $\sigma : X^+ \rightarrow X/S$ which sends C_1^+ to C_1 and C_2' to C_2 . Furthermore, there exists an automorphism $\iota : X \rightarrow X/S$ which exchanges the two curves C_1 and C_2 .*

Proof. The arguments here are due to Kawamata [7, Example 3.8(2)]. We make some aspects of the proof explicit by working with local defining equations given by Reid [11]. In what follows, we use the notation $\bar{\cdot}$ to denote objects on a family $\bar{\pi} : \bar{X} \rightarrow \bar{S}$ over an affine base, while objects with no bar will be the restrictions to a certain germ.

Let $\bar{S} = \mathbb{A}^2$, with coordinates t_1 and t_2 . Fix two distinct complex numbers a_1 and a_2 and define $\bar{X}_0 \subset (\mathbb{A}^1 \times \mathbb{A}^1) \times \bar{S}$ by the equation

$$x_1^2 = ((x_2 - a_1)^2 - t_1)((x_2 - a_2)^2 - t_2).$$

The closure $\bar{X} \subset (\mathbb{P}^1 \times \mathbb{P}^1) \times \bar{S}$ is smooth, and the second projection $\bar{\pi} : \bar{X} \rightarrow \bar{S}$ is proper. The fiber of $\bar{\pi}$ over a general point (t_1, t_2) is a smooth curve of genus 1. If exactly one of t_1 and t_2 is zero, the fiber is nodal, while if $t_1 = t_2 = 0$, the fiber is given by $x_1^2 = (x_2 - a_1)^2(x_2 - a_2)^2$. This central fiber has two components, the rational curves C_1 defined by $x_1 = -(x_2 - a_1)(x_2 - a_2)$ and C_2 defined by $x_1 = (x_2 - a_1)(x_2 - a_2)$. The restriction of $\bar{\pi} : \bar{X} \rightarrow \bar{S}$ to the germ at $(0, 0) \in \bar{S}$ is the versal deformation space $\pi : X \rightarrow S$ considered above. The involution $\iota : \bar{X} \rightarrow \bar{X}/\bar{S}$ defined by $\iota(x_1, x_2) = (-x_1, x_2)$ exchanges the two components of the central fiber.

There is a section $\bar{\sigma} : \bar{S} \rightarrow \bar{X}$ given by

$$\begin{aligned} x_2(t_1, t_2) &= \frac{a_1 + a_2}{2} - \frac{t_1 - t_2}{2(a_1 - a_2)}, \\ x_1(t_1, t_2) &= (x_2(t_1, t_2) - a_1)^2 - t_1. \end{aligned}$$

This has $\bar{\sigma}(0, 0) = \left(\frac{(a_2 - a_1)^2}{4}, \frac{1}{2}(a_1 + a_2) \right)$, which lies on C_1 and is disjoint from C_2 .

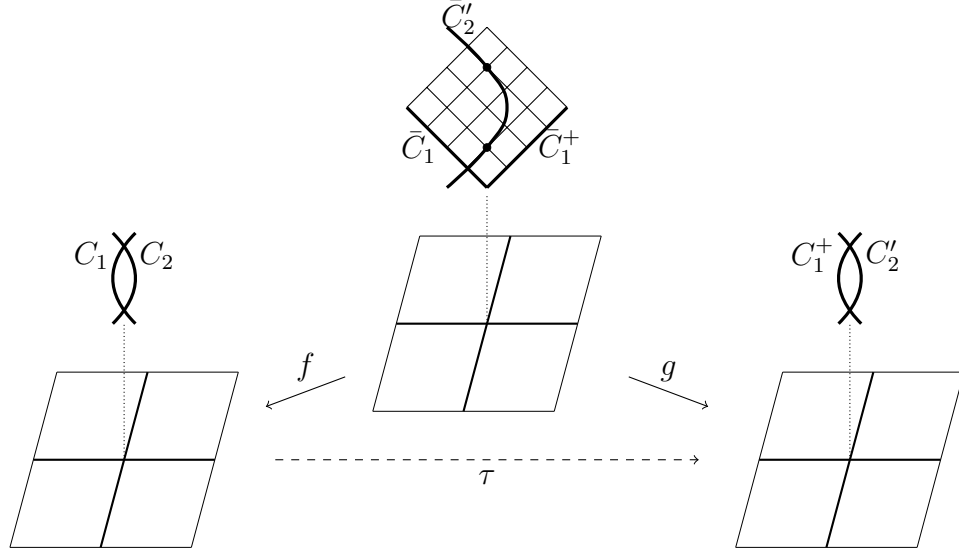
Let $\bar{\Sigma}_1$ be the divisor $\sigma(\bar{S})$. Since $\bar{\Sigma}_1 \cdot C_1 = 1$ and $\bar{\Sigma}_1 \cdot C_2 = 0$, the curves C_1 and C_2 have distinct classes in $N_1(\bar{X}/\bar{S})$. Since all other fibers of $\bar{\pi}$ are irreducible, it must be that $N^1(\bar{X}/\bar{S})$ has dimension 2. The divisor $2\iota_*(\bar{\Sigma}_1) - \bar{\Sigma}_1$ has positive degree on general fibers, and so is $\bar{\pi}$ -big. Since \bar{S} is affine, there is an effective divisor $\bar{\Delta}$ representing this class. For sufficiently small ϵ , the pair $(\bar{X}, \epsilon\bar{\Delta})$ is klt. Since $\bar{\Delta} \cdot C_1 < 0$, there exists a $(K_{\bar{X}/\bar{S}} + \epsilon\bar{\Delta})$ -flip $\tau : \bar{X} \dashrightarrow \bar{X}^+$, which is a $K_{\bar{X}/\bar{S}}$ -flop. The map $\bar{\pi}^+ : \bar{X}^+ \rightarrow \bar{S}$ is a minimal model of \bar{X}^+ . The strict transform of $\bar{\Sigma}_1$ on \bar{X}^+ is smooth, contains the curve C_1^+ , and satisfies $\tau_*\bar{\Sigma}_1 \cdot C_2' = 2$.

Since $\pi : X \rightarrow S$ is a versal deformation space and $\pi^+ : X^+ \rightarrow S$ has the same local structure, there exists an isomorphism $\beta : X^+ \rightarrow X$ over S . However, this map might not be defined over the identity map on S . The divisor $\Sigma_2 = \beta_*(\tau_*(\Sigma_1))$ is a smooth divisor on X , containing C_1 , and meeting C_2 at two points. There is a translation on the smooth fibers of π sending Σ_1 to Σ_2 , which defines a birational automorphism $\gamma : X \dashrightarrow X$ over the identity on S . The map $\pi \circ \gamma : X \rightarrow S$ must be isomorphic to some minimal model of X over S , and indeed must be isomorphic to $\pi^+ : X^+ \rightarrow S$ since the strict transforms of Σ_1 under γ and τ have the same numerical classes. It follows that there exists an isomorphism $\sigma : X^+ \rightarrow X$ over the identity of S . Replacing σ with $\sigma \circ \iota$ if necessary, we may assume that $\sigma(C_1^+) = C_1$ and $\sigma(C_2') = C_2$, as required. \square

Each of the maps $\sigma \circ \tau$ and ι is a birational involution of X over S , but we will soon see that the composition $\phi = (\sigma \circ \tau) \circ \iota$ is of infinite order. Since $\iota(C_2) = C_1$, the effect of repeatedly applying ϕ is to flop C_1 , then C_2 , then C_1 again, and so on. We will denote by ϕ_*D the strict transform of a divisor D under a birational map ϕ , and use the same notation for the induced map on numerical groups when confusion seems unlikely.

To an effective divisor D on X , associate the 4-tuples

$$\begin{aligned} v_D &= (D \cdot C_1, D \cdot C_2, \text{mult}_{C_1}(D), \text{mult}_{C_2}(D)), \\ \sigma_D &= (D \cdot C_1, D \cdot C_2, \sigma_{C_1}(D; X/S), \sigma_{C_2}(D; X/S)). \end{aligned}$$

FIGURE 2. Resolution of the flop τ

Lemma 8. Suppose that D is a divisor on X , and let \tilde{D} denote the strict transform of D under the flop $\tau : X \dashrightarrow X^+$. Then

- (1) $\tilde{D} \cdot C_1^+ = -D \cdot C_1$,
- (2) $\tilde{D} \cdot C_2' = D \cdot C_2 + 2(D \cdot C_1)$,
- (3) $\text{mult}_{C_1^+}(\tilde{D}) = \text{mult}_{C_1}(D) + D \cdot C_1$,
- (4) $\text{mult}_{C_2'}(\tilde{D}) = \text{mult}_{C_2}(D)$.

In matrix form, we have $v_{\phi_* D} = M v_D$ and $\sigma_{\phi_* D} = M \sigma_D$ where

$$M = \begin{pmatrix} 2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Proof. Let W be the graph of the flop τ :

$$\begin{array}{ccc} & W & \\ f \swarrow & & \searrow g \\ X & \dashrightarrow^{\tau} & X^+ \end{array}$$

Since τ is the flop of a rational curve with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, there is a single f -exceptional divisor E on W , which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and has normal bundle of bidegree $(-1, -1)$. Let \bar{C}_1 be a ruling of E contracted by g , so that f sends \bar{C}_1 isomorphically to C_1 . Similarly, let \bar{C}_1^+ be a ruling of E contracted by f , so that g maps \bar{C}_1^+ isomorphically onto C_1^+ . Lastly, let \bar{C}_2' be the strict transform of C_2 on W , a curve which meets E transversely at 2 points. Then write

$$f^* D + aE = g^* \tilde{D},$$

for some constant a . Taking the intersection of both sides with \bar{C}_1 yields $D \cdot C_1 + a(E \cdot \bar{C}_1) = 0$. Since $E \cdot \bar{C}_1 = -1$, we obtain $a = D \cdot C_1$. Intersecting with \bar{C}_1^+ , we have $-a = \tilde{D} \cdot C_1^+$.

Similarly, intersecting with \bar{C}'_2 , we have $D \cdot C_2 + a(E \cdot \bar{C}'_2) = \tilde{D} \cdot C'_2$, and since $E \cdot \bar{C}'_2 = 2$, we have (2). It is clear that $\text{mult}_{C'_2}(\tilde{D}) = \text{mult}_{C_2}(D)$, since τ is an isomorphism at the generic point of C_2 . Finally,

$$\text{mult}_{C_1^+}(\tilde{D}) = \text{mult}_E(g^*\tilde{D}) = \text{mult}_E(f^*D) + a = \text{mult}_{C_1}(D) + a.$$

These calculations immediately yield $v_{\phi_*D} = Mv_D$, since the second map ι exchanges the two curves C_1 and C_2 . Write D_m for a general divisor linearly equivalent to mD , and then

$$\begin{aligned} \sigma_{C_1}(\phi_*D) &= \lim_{m \rightarrow \infty} \frac{1}{m} \text{mult}_{C_1}(\phi_*D_m) = \lim_{m \rightarrow \infty} \frac{1}{m} (\text{mult}_{C_1} D_m + D_m \cdot C_1) \\ &= \left(\lim_{m \rightarrow \infty} \frac{1}{m} \text{mult}_{C_1} D_m \right) + D \cdot C_1 = \sigma_{C_1}(D) + a. \end{aligned} \quad \square$$

We are now in position to make the main computation.

Theorem 9. *Let $\pi : X \rightarrow S$ be the versal deformation space of a singular fiber of Kodaira type I_2 , and let C_1 be a component of the central fiber. Suppose that D is a divisor on the boundary of the cone $\overline{\text{Eff}}(X/S)$. Then $\sigma_{C_1}(D; X/S) = \infty$.*

Proof. Fix a π -ample effective \mathbb{Q} -divisor $H = H_0$ on X with $H \cdot C_1 = H \cdot C_2 = 1$ and $\text{mult}_{C_i}(H) = 0$. Let $H_n = \phi_*^n(H)$ be the strict transform of H on X under n applications of ϕ . Using the Jordan decomposition of M , which has a 3×3 block associated to the eigenvalue 1, we compute $\sigma_{H_n} = (2n+1, -2n+1, n(n-1)/2, n(n+1)/2)$:

n	$H_n \cdot C_1$	$H_n \cdot C_2$	$\text{mult}_{C_1} H_n$	$\text{mult}_{C_2} H_n$
0	1	1	0	0
1	3	-1	0	1
2	5	-3	1	3
3	7	-5	3	6
		...		
n	$2n+1$	$-2n+1$	$\frac{n(n-1)}{2}$	$\frac{n(n+1)}{2}$

The key feature of the example is that while $H_n \cdot C_1$ grows linearly in n , the multiplicity $\text{mult}_{C_1}(H_n)$ grows quadratically. Let D be the divisor class on the boundary of $\overline{\text{Eff}}(X/S)$ with $D \cdot C_1 = 1$ and $D \cdot C_2 = -1$. Since C_1 and C_2 span $N_1(X/S)$, we see that

$$H_n \equiv_{\pi} (2n)D + H_0.$$

It follows that $\frac{1}{2n}H_n \equiv_{\pi} D + \frac{1}{2n}H_0$ is a sequence of divisors converging to D , whose multiplicities along the curves is known. By Definition 1, we compute

$$\sigma_{C_1}(D; X/S) = \lim_{n \rightarrow \infty} \text{mult}_{C_1}(D + \frac{1}{2n}H_0) = \lim_{n \rightarrow \infty} \frac{1}{2n} \text{mult}_{C_1} H_n = \lim_{n \rightarrow \infty} \frac{n-1}{4} = \infty. \quad \square$$

Note that $\text{codim } \pi(C_1) = 2$, so there is no contradiction with Theorem 4(3).

Corollary 10. *If $f : W \rightarrow X$ is the blow-up along C_1 with exceptional divisor E , then $\tilde{D} = f^*D$ has $\sigma_E(\tilde{D}; W/S) = \infty$ and $N_{\sigma}(\tilde{D}; W/S)$ contains the divisor E with infinite coefficient. In particular, there does not exist a birational model $g : Z \rightarrow W$ for which $g^*\tilde{D}$ admits a decomposition $g^*\tilde{D} = P + N$ with P a $g \circ (f \circ \pi)$ -movable divisor and N effective.*

Proof. By Theorem 5(2), if $f : W \rightarrow X$ is the blow-up along C_1 , with exceptional divisor E , we have $\sigma_E(f^*D; W/S) = \infty$. Now, suppose that $g : Y \rightarrow W$ is any birational map, and that $g^*f^*D = P + N$, where P is a $(g \circ f \circ \pi)$ -movable divisor and N is effective. Let \tilde{E} denote the strict transform of E on Y . Then

$$\sigma_{\tilde{E}}(g^*f^*D; Y/S) \leq \sigma_{\tilde{E}}(P; Y/S) + \sigma_{\tilde{E}}(N; Y/S) = \sigma_{\tilde{E}}(N; Y/S).$$

The last of these is finite since N is effective, while the first is infinite, a contradiction. This completes the proof. \square

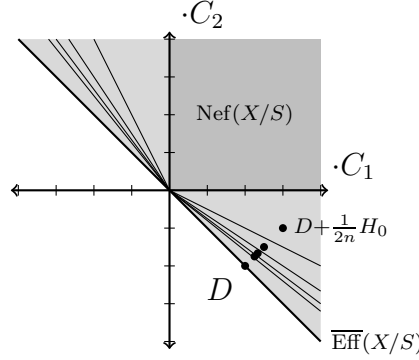


FIGURE 3. Chambers in $N^1(X/S)$

Remark 1. The construction of the automorphism $\beta : X^+ \rightarrow X/S$ in Lemma 7 (from [7]) is only over a surface germ S , for it relies on the fact that $\pi : X \rightarrow S$ is a versal deformation space. However, the local analytic results of Theorem 9 and Corollary 10 imply that the same pathological behavior occurs even when the base S is an affine surface. We have seen that there is a projective family $\bar{\pi} : \bar{X} \rightarrow \bar{S}$ where \bar{S} is an affine surface, such that the restriction of $\bar{\pi}$ to the germ at a point $0 \in \bar{S}$ coincides with the map $\pi : X \rightarrow S$.

If \bar{G} is a $\bar{\pi}$ -big divisor, with restriction G to the germ, then $\sigma_{C_1}(\bar{G}; \bar{X}/\bar{S}) \geq \sigma_{C_1}(G; X/S)$: indeed, if \bar{G}' is an effective divisor on \bar{X} which is $\bar{\pi}$ -numerically equivalent to \bar{G} , its restriction to the central germ is an effective divisor on X which is π -numerically equivalent to G . Thus the infimum defining $\sigma_{C_1}(\bar{G}; \bar{X}/\bar{S})$ is taken over a subset of the infimum defining $\sigma_{C_1}(G; X/S)$ in Definition 1, giving the claimed inequality. It follows that in the limit at the pseudoeffective boundary, $\sigma_{\bar{C}_1}(\bar{D}; \bar{X}/\bar{S}) \geq \sigma_{C_1}(D; X/S)$, and it must be that $\sigma_{\bar{C}_1}(\bar{D}; \bar{X}/\bar{S})$ is infinite as well. The claims about Zariski decomposition follow as before.

4. A GENERAL SET-UP

The key feature that made possible the computation of the preceding example is that if the four numbers $D \cdot C_i$ and $\text{mult}_{C_i}(D)$ are all known, then the same four invariants can be computed for the strict transform of D under ϕ using Lemma 8. In this section, we give an explanation for this, and describe how to make analogous computations in a more general setting.

Suppose that $\phi : X \dashrightarrow X$ is a pseudoautomorphism over S , i.e. a birational map for which neither ϕ nor ϕ^{-1} contracts any divisors. We will say that a birational morphism $f : Y \rightarrow X$

from a normal \mathbb{Q} -factorial variety Y is a *small lift* of ϕ if the induced map $\psi : Y \dashrightarrow Y$ is also a pseudoautomorphism.

$$\begin{array}{ccc} Y & \xrightarrow{\psi} & Y \\ f \downarrow & & \downarrow f \\ X & \xrightarrow{\phi} & X \end{array}$$

Observe that if $f : Y \rightarrow X$ is a small lift, then the map ψ must permute the exceptional divisors of f .

Example 11. Suppose that $\phi : X \dashrightarrow X$ is a pseudoautomorphism and x is a point not contained in $\text{indet } \phi$. The blow-up $f : \text{Bl}_x X \rightarrow X$ is a small lift of ϕ if and only if x is a fixed point of ϕ . If x is not a fixed point, then the induced map $\psi : Y \dashrightarrow Y$ contracts the exceptional divisor E , while if x is fixed, then $\psi|_E : E \rightarrow E$ is an automorphism.

The more interesting examples are those in which f contracts a divisor lying over $\text{indet } \phi$.

Example 12. Next we construct a small lift of the map $\phi : X \dashrightarrow X/S$ from Section 3. Let $f : W \rightarrow X$ be the blow-up along C_1 as before, with exceptional divisor E_1 , and let $h : Y \rightarrow W$ be the blow-up along \tilde{C}'_2 , with exceptional divisor E_2 . The two exceptional divisors E_1 and E_2 are swapped by the induced map $\psi : Y \dashrightarrow Y$, and $h \circ f$ is a small lift.

$$\begin{array}{ccccc} Y & \xrightarrow{\psi} & & & Y \\ & \searrow h & & & \downarrow h \circ f \\ h \circ f \downarrow & & W & & \downarrow h \circ f \\ & \nearrow f & & \searrow g & \\ X & \xrightarrow{\phi} & & & X \end{array}$$

The curves C_1 and C_2 could have been blown up in the opposite order, yielding a different small lift $f' : Y' \rightarrow X$. This makes no real difference: the threefolds Y and Y' differ only by flops, and strict transform induces an identification $N^1(Y) \xrightarrow{\sim} N^1(Y')$ with respect to which the maps ψ_* and ψ'_* coincide.

If $f : Y \rightarrow X$ is a small lift, it follows from the negativity lemma [1, Lemma 3.6.2] that there is a decomposition $N^1(Y) = f^*N^1(X) \oplus V_E$, where $V_E = \bigoplus_i \mathbb{R} \cdot [E_i]$. If D is a divisor class on X , it is not necessarily true that $f^*\phi_*D = \psi_*f^*D$. However, the difference $f^*\phi_*D - \psi_*f^*D$ is an f -exceptional divisor, since

$$f_*(f^*\phi_*D - \psi_*f^*D) = \phi_*D - f_*\psi_*f^*D = \phi_*D - \phi_*f_*f^*D = \phi_*D - \phi_*D = 0.$$

Define $K : N^1(X) \rightarrow V_E$ by $K = f^*\phi_* - \psi_*f^*$. The next lemma characterizes the action of the strict transform $\psi_* : N^1(Y) \rightarrow N^1(Y)$ with respect to this decomposition.

Lemma 13. Suppose that $f : Y \rightarrow X$ is a small lift of a pseudoautomorphism $\phi : X \dashrightarrow X$. With respect to the decomposition $N^1(Y) \cong f^*N^1(X) \oplus V_E$, ψ_* is given in block form as

$$\psi_* = \left(\begin{array}{c|c} \phi_* & 0 \\ \hline -K & P \end{array} \right),$$

where P is the permutation matrix for the action of ψ_* on the E_i . The eigenvalues of ψ_* are the union of those of ϕ_* and those of P , which are roots of unity. Its eigenvectors are

- (1) $f^*v_i - (\lambda I - P)^{-1}Kv_i$, where v_i are the eigenvectors of ϕ_* , with eigenvalues λ_i ;
- (2) E_i , the exceptional divisors of f , with eigenvalues that are roots of unity.

Proof. For a divisor D on X , $\psi_*f^*D = f^*\phi_*D - KD$, while the exceptional divisors E_i are simply permuted by ψ ; this gives the block form of the map. The eigenvectors follow from elementary linear algebra. \square

A rational map $\phi : X \dashrightarrow Y$ is said to be D -non-negative for an \mathbb{R} -divisor D if on some common resolution $f : W \rightarrow X$, $g : W \rightarrow Y$, we have $f^*D + E = g^*(\phi_*D)$, where E is an effective g -exceptional divisor. If $\phi : X \dashrightarrow X$ is a pseudoautomorphism with a small lift f , then we may consider a resolution of the form

$$\begin{array}{ccc}
 & W & \\
 p \swarrow & & \searrow q \\
 Y & \text{---} \text{---} \text{---} \text{---} & Y \\
 f \downarrow & \psi & \downarrow f \\
 X & \text{---} \text{---} \text{---} \text{---} & X \\
 & \phi &
 \end{array}$$

If D is a divisor on X for which ϕ is D -non-negative, then we have $p^*f^*D + E = q^*f^*\phi_*D$ with $E \geq 0$. Pushing forward both sides by q , this gives

$$\begin{aligned}
 q_*p^*f^*D + q_*E &= f^*\phi_*D \\
 \psi_*f^*D + E' &= f^*\phi_*D,
 \end{aligned}$$

where E' is an effective f -exceptional divisor. In particular, $KD = f^*\phi_*D - \psi_*f^*D = E'$ is effective.

Next we observe that if the divisorial Zariski decomposition $P_\sigma(f^*D)$ is known for some divisor D , the decomposition $P_\sigma(f^*\phi_*D)$ can often be computed, using the strict transform under $\psi : Y \dashrightarrow Y$. For simplicity, we assume that ψ fixes each of the f -exceptional divisors E_i ; this can always be arranged by replacing ϕ by a suitable iterate. This assumption implies that the permutation matrix P is the identity, and that $\psi_*(KD) = KD$ since KD is exceptional.

Lemma 14. *Suppose $\phi : X \dashrightarrow X$ is a pseudoautomorphism over S , and that D is a class in $N^1(X/S)$. Then $N_\sigma(\phi_*D; X/S) = \phi_*N_\sigma(D; X/S)$. If $N_\sigma(D; X/S)$ is finite, then $P_\sigma(\phi_*D; X/S) = \phi_*P_\sigma(D; X/S)$ as well. If ϕ is D -non-negative and $N_\sigma(f^*D; Y/S)$ is finite, then $P_\sigma(f^*\phi_*D; Y/S) = \psi_*P_\sigma(f^*D; Y/S)$.*

Proof. Since ϕ neither contracts nor extracts any divisors, for any prime divisor E we have $\sigma_E(D; X/S) = \sigma_{\phi_*E}(\phi_*D; X/S)$. The claim for $N_\sigma(\phi_*D; X/S)$ follows, and that for $P_\sigma(\phi_*D; X/S)$ is immediate.

Now, by the D -non-negativity hypothesis on ϕ , KD is an effective exceptional divisor. By [10, Lemma 3.5.1], if E is an effective exceptional divisor, we have $N_\sigma(f^*D + E) = N_\sigma(f^*D) + E$. This means that

$$\begin{aligned}
 N_\sigma(f^*\phi_*D) &= N_\sigma(\psi_*f^*D + KD) = N_\sigma(\psi_*(f^*D + KD)) = \psi_*N_\sigma(f^*D + KD) \\
 &= \psi_*N_\sigma(f^*D) + \psi_*KD = \psi_*N_\sigma(f^*D) + KD.
 \end{aligned}$$

We have made use of the fact that E is effective by the non-negativity hypothesis on D . It is now simple to compute the positive part of the decomposition:

$$\begin{aligned} P_\sigma(f^*\phi_*D) &= f^*\phi_*D - N_\sigma(f^*\phi_*D) = f^*\phi_*D - \psi_*N_\sigma(f^*D) - KD \\ &= \psi_*f^*D - N_\sigma(\psi_*f^*D) = P_\sigma(\psi_*f^*D) = \psi_*P_\sigma(f^*D). \end{aligned} \quad \square$$

Remark 2. The example of Section 3 can be interpreted as an instance of the calculations in this section. A small lift of the map ϕ is constructed in Example 12. Let F_1, F_2 be a basis for $N^1(X/S)$ dual to C_1 and C_2 . A basis for $N^1(Y/S)$ is given by the four classes $(h \circ f)^*F_1$, $(h \circ f)^*F_2$, E_1 , and E_2 . The vector v_D gives the coefficients for the class of the strict transform of D on Y with respect to this above basis. Lemma 8 is nothing more than the calculation of the induced map ψ_* of Lemma 13. The final calculation in Theorem 9 can then be carried out as a repeated application of Lemma 14.

Suppose now that $S = \text{Spec } \mathbb{C}$ and $\phi : X \dashrightarrow X$ is a pseudoautomorphism whose action on $N^1(X)$ has a unique largest eigenvalue, greater than 1, and that $f : Y \rightarrow X$ is a small lift of ϕ . We are then able to compute the Zariski decomposition of the divisor f^*D_ϕ using the above result.

Corollary 15. *Let D_ϕ be the dominant eigenvector of $\phi_* : N^1(X) \rightarrow N^1(X)$, and D_ψ be the dominant eigenvector of $\psi_* : N^1(Y) \rightarrow N^1(Y)$. Then $P_\sigma(f^*D_\phi) = D_\psi$.*

Proof. If D is any pseudoeffective divisor on X , then for every n we have

$$P_\sigma(f^*(\lambda^{-n}\phi_*^n D)) = \lambda^{-n}\psi_*^n P_\sigma(f^*D).$$

Take $D = D_\phi + D_{\phi^{-1}}$, so that the above reduces to

$$P_\sigma(f^*(D_\phi + \lambda^{-2n}D_{\phi^{-1}})) = \lambda^{-n}\psi_*^n P_\sigma(f^*D).$$

The left hand side converges to $P_\sigma(f^*D_\phi)$ by Proposition 5(1). With a suitable choice of scaling, the right hand side converges to D_ψ . \square

5. ACKNOWLEDGEMENTS

I am grateful to James McKernan for some useful questions and suggestions. This material is based upon work supported by the National Science Foundation under agreement No. DMS-1128155. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

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